SOME EVIDENCE FOR THE COLEMAN-OORT CONJECTURE

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Abstract. The Coleman-Oort conjecture says that for large g there are no positive-dimensional Shimura subvarieties of A_g generically contained in the Jacobian locus. Counterexamples are known for $g \leq 7$. They can all be constructed using families of Galois coverings of curves satisfying a numerical condition. These families are already classified in cases where: a) the Galois group is cyclic, b) it is abelian and the family is 1-dimensional, and c) $g \leq 9$. By means of carefully designed computations and theoretical arguments excluding a large number of cases we are able to prove that for $g \leq 100$ there are no other families than those already known.

1. Introduction

1.1. Denote by A_g the moduli space of principally polarized complex abelian varieties of dimension g, by M_g the moduli space of smooth complex algebraic curves of genus g and by $j\colon \mathsf{M}_g \to \mathsf{A}_g$ the period mapping (or Torelli mapping), which associated to $[C] \in \mathsf{M}_g$ the moduli point of the Jacobian variety JC provided with the theta polarization. The *Jacobian locus* is the image $j(\mathsf{M}_g)$. By $\overline{j(\mathsf{M}_g)}$ we denote the closure of $j(\mathsf{M}_g)$ in A_g.

On A_g there is a tautological $\mathbb{Q}$-variation of the Hodge structure (in the orbifold sense): if A is a principally polarized abelian variety, the fibre over its moduli point $[A] \in \mathsf{A}_g$ is $H^1(A, \mathbb{Q})$ with its Hodge structure of weight 1. In general, given a variation of the Hodge structure $H \to B$, it is interesting to consider the points $b \in B$ where the Hodge structure is "more symmetric" than over the general point. Making precise the meaning of "more symmetric" requires some effort. In the simplest case this means that the Hodge structure has more automorphism than usual. For example for the variation over A_1, the general point has no automorphisms beyond $\{\pm 1\}$, while the points with more automorphisms represent the well-known elliptic curves with automorphisms $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. The general case is more complicated since the symmetry is not at the level of automorphisms but is detected by Hodge classes in general tensor spaces. The loci obtained in this way are

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called the *Hodge loci* of the variation of the Hodge structure. In the case of A_g they are also called *special subvarieties* or *Shimura subvarieties.* (See [26, §3.3] and [16].) A subvariety $Z \subset \mathsf{A}_g$ is said to be *generically contained* in $j(\mathsf{M}_g)$ if $Z \subset \overline{j(\mathsf{M}_g)}$ and $Z \cap j(\mathsf{M}_g) \neq \emptyset$. Arithmetical considerations led first Coleman and later Oort [27] to the following

Conjecture 1.2 (Coleman-Oort). *For large g there are no special subvarieties of positive dimension generically contained in $j(\mathsf{M}_g)$.*

(See [26, §4] for more details.) This expectation is also motivated by another stronger expectation originating from the point of view of differential geometry: special subvarieties are totally geodesic with respect to the locally symmetric (orbifold) metric on A_g (the one coming from the Siegel space). If one believes that $j(\mathsf{M}_g)$ bears no strong relation to the ambient geometry of A_g, in particular that it is very curved inside A_g, then it is natural to expect that $j(\mathsf{M}_g)$ contains generically no totally geodesic subvarieties, and in particular no Shimura subvarieties (see [8], [17], [15] for results in this direction).

What makes the problem more interesting is that for low genus examples of such Shimura varieties generically contained in $j(\mathsf{M}_g)$ do exist! All the examples known so far are in genus $g \leq 7$ and arise from one of the following two constructions.

1.3. **First construction**. Let G be a finite group acting on a curve C. Consider the family of curves $\mathscr{C} \to B$ with a G-action of the same topological type (see below for the precise definition). For every m, $H^0(C_b, mK_{C_b})$ is a representation of G and its equivalence class is independent of $b \in B$. Denote by $B' \subset \mathsf{M}_g$ the moduli image of B and by Z the closure of $j(B')$ in A_g. In [12, 13] it is proven that if

$$(*) \qquad \dim(S^2(H^0(K_{C_b})))^G = \dim H^0(2K_{C_b})^G,$$

then Z is a Shimura variety generically contained in $j(\mathsf{M}_g)$. We also say that the family of G-covers $\mathscr{C} \to B$ *yields* a Shimura variety to mean that Z is Shimura. We refer to such a Shimura variety as a *counter-example* to Coleman-Oort conjecture. Several counter-examples are known, see Theorem 1.5 below.

1.4. **Second construction**. Consider a Shimura variety Z generically contained in $j(\mathsf{M}_g)$ obtained as in 1.3 from a family of G-curves $\mathscr{C} \to B$. Denote by g' the genus of C_b/G. Let $\mathrm{Nm} : JC_b \to J(C_b/G)$ be the norm map of the covering $f_b : C_b \to C_b/G$, defined by $\mathrm{Nm}(\sum_i p_i) := \sum_i f_b(p_i)$, Then $(\ker \mathrm{Nm})^0 \subset JC_b$ is an abelian subvariety, the generalized Prym variety of the covering f_b. The theta polarization of JC_b restricts to a polarization of some type δ on the Prym variety. We get maps

$$\varphi : B \longrightarrow \mathsf{M}_g, \quad \varphi(b) := [C_b/G],$$
$$\mathscr{P} : B \longrightarrow \mathsf{A}^\delta_{g-g'}, \quad \mathscr{P}(b) := [(\ker \mathrm{Nm})^0].$$

$\mathscr{P}$ is the *generalized Prym map.* If $g' = 0$ the map φ is of course constant, $\mathsf{A}^\delta_{g-g'} = \mathsf{A}_g$ and $\mathscr{P}$ is just the Torelli map, so we get nothing new. If instead

$g' > 0$, the irreducible components of the fibres of $\mathscr{P}$ and φ are totally geodesic subvarieties and countably many of them are in fact Shimura, see [20] and [14, Thm. 3.9, Thm. 3.11]. Thus for $g' > 0$ this construction gives uncountably many totally geodesic non-Shimura varieties generically contained in $j(\mathsf{M}_g)$ and countably many Shimura varieties generically contained in $j(\mathsf{M}_g)$.

Let us summarize what is known about the counter-examples obtained via these constructions.

Theorem 1.5.

a) There are 38 families of Galois coverings of the projective line satisfying $(*)$ *with* $2 \leq g \leq 7$. *For* $g \leq 9$ *there are no other counter-examples. See* [30, 25, 26, 12].

b) There are 6 families of Galois coverings of elliptic curves satisfying $(*)$ *with* $2 \leq g \leq 4$. *For* $g \leq 9$ *there are no other counter-examples. See* [13].

c) If a family satisfies $(*)$ *and* $g' > 0$, *then necessarily* $g' = 1$ *and the family is one of those in (b). See* [14].

1.6. Note that we focus on $g \geq 2$, since for $g = 1$ there are infinitely many 1-dimensional families satisfying $(*)$.

In fact, for every elliptic curve C the involution $p \mapsto -p$ acts trivially on both $S^2H^0(K_C)$ and $H^0(2K_C)$. Let G be the group of the biholomorphisms of C generated by it and by a finite group of translations. Then $S^2H^0(K_C)^G = S^2H^0(K_C) \cong \mathbb{C} \cong H^0(2K_C) = H^0(2K_C)^G$, so giving examples of $(*)$ with G of order arbitrarily high. Two of these families are listed in Table 2 in [12].

However all these families are irrelevant for the Coleman-Oort conjecture, since in all cases $B' = \mathsf{M}_1$. Note also that some of the families of Theorem 1.5 yield the same Shimura variety, i.e. have the same image in moduli, see [12, 13].

1.7. It follows from Theorem 1.5 (c) that all the cases where $(*)$ holds and $g' > 0$ are already known and also that no new examples can be found using the second construction 1.4. Therefore, in order to construct new examples using the two methods above (or to exclude the existence of such examples) we can restrict to the first construction with $g' = 0$, i.e. $C_b/G = \mathbb{P}^1$.

The purpose of this paper is to provide further evidence for the Coleman-Oort conjecture, employing a computational approach complemented by theoretical arguments. Our result is the following improvement of Theorem 1.5.

Theorem 1.8. *The positive-dimensional families of Galois covers satisfying* $(*)$ *with* $2 \leq g \leq 100$ *are only those of Theorem 1.5.*

1.9. The fact that we found **no** new families at all is strong evidence that there are no more families satisfying $(*)$. Since all known counter-examples to the Coleman-Oort conjecture can be constructed using these families, this also suggests that either further counter-examples do not exist or they are of a completely different nature.

1.10. Families of G-covers are identified by data of combinatorial and group-theoretical nature. We explain this in §2. So the basic strategy is obviously to list all these data and check condition $(*)$ for each datum in the list. Since the list of these data is extremely long, one needs to avoid unnecessary computations. The first observation is that many data give rise to the same family. More precisely call two data Δ and Δ' *Hurwitz equivalent* if they have the same group G and if the families corresponding to them are isomorphic as families of algebraic curves with G-action. It turns out that Hurwitz equivalence classes can be huge. To check condition $(*)$ for all the families of some genus, one would start by choosing a representative out of any Hurwitz equivalence class, and proceed by checking $(*)$ for all the representatives. However, the identification of a single representative inside each class is a daunting task, since the classes are huge and Hurwitz equivalence is rather complicated. (An algorithm dealing with Hurwitz equivalence appears in [2]. It was used in [12] and [13]. An improvement of this algorithm is given in [3]. We hope to address the problem of algorithmic computation of Hurwitz equivalence in future work.)

Luckily there is another equivalence relation on data, much coarser than the Hurwitz equivalence, which is appropriate to our problem: if $\Delta = (G, g_1, \ldots, g_r)$, then the number $N = N(\Delta)$ only depends on the conjugacy classes $C_1 = [g_1], \ldots, C_r = [g_r]$. Also the order of these is completely irrelevant. The unordered sequence $(C_1, \ldots, C_r)$ is called a *refined passport*. (See Definition 3.7.) So our problem depends only on refined passports, more precisely on their $\mathrm{Aut}(G)$-orbits, which are considerably less in number than Hurwitz equivalence classes, leading to much shorter execution times. Notice that in some cases refined passports (even if taken up to the action of $\mathrm{Aut}(G)$) are still too many to be stored simultaneously into memory, but this is not a problem, since we only need to perform an iteration to check $(*)$ on each individually.

Even after this great simplification the computation remains quite formidable, at least for the computers at our disposal. We use a number of tricks to reduce the data that must be considered. Several exclusions (e.g. cyclic groups) follow from previous results (see Theorem 3.3). We complement them with Corollary 3.13, which effectively eliminates more than 90% of the data, including some of the hardest cases, thus allowing us to complete the computation.

1.11. For the implementation of the algorithm we used `MAGMA` [23], which is quite suited to the task at hand since it allows working with groups, group actions and representations, in particular computing characters, orbits and stabilizers; furthermore, it contains a database of groups of small order. Our code is available at [9].

The problem lends itself easily to parallelization, since each group and signature is treated independently; however, `MAGMA` does not support parallelization natively. The first part of the computation (Algorithm 1) was parallelized using the standard tool [32]. On the other hand, the rest of the computation can become quite memory-intensive; this leads to technical difficulties, mainly concerning situations in which one of the processes is terminated for lack of memory, which were addressed by writing an ad hoc external program to run the `MAGMA` script.

Using a computer with 56 Intel Xeon 2.60GHz CPU and 128 GB of RAM we were able to finish the computations in less than three days.

1.12. An important point to stress is the following. Condition $(*)$ is sufficient for a family to yield a Shimura variety. In general it is unknown if it is also necessary. In this paper we only check whether condition $(*)$ holds. So we cannot exclude that these families give rise to counter-examples to Coleman-Oort conjecture.

1.13. The plan of the paper is as follows. In §2 we recall the description of the families of G-curves and some basic facts concerning the multiplication map on sections of the canonical bundle, which is related with condition $(*)$. At the end we prove Lemma 2.17, which deals with the behaviour of condition $(*)$ when passing from a given family to a quotient by a normal subgroup. In §3 we gather several facts of quite different nature, some well-known, some new, which we have found useful to rule out several cases. This has been essential in order to complete the computation. Finally §4 contains a thorough explanation of the algorithm.

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2. Families of G-curves

2.1. The purpose of this section is to describe some group-theoretic and combinatorial data from which one can construct algebraic families of curves with prescribed symmetry. We will denote by Δ the datum and by $\mathscr{C}_\Delta \to B_\Delta$ the corresponding family of curves. The image of B_Δ in M_g will be denoted by M_Δ. We are interested in the closure of M_Δ in A_g. As explained in 1.3, when $(*)$ holds this closure is a Shimura variety generically contained in the Jacobian locus. This is explained in more detail at the end of this section, together with some related remarks on the multiplication map.

In the following, unless otherwise stated, we assume that the genus is at least 2. For $r \geq 3$, set

$$\Gamma_r := \langle \gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^r \gamma_i = 1 \rangle.$$

Definition 2.2. *If G is a finite group an epimorphism $\theta : \Gamma_r \to G$ is called* admissible *if $\theta(\gamma_i) \neq 1$ for $i = 1, \ldots, r$. An r-datum is a pair $\Delta = (G, \theta)$ where $G \in \mathfrak{G}$ and $\theta : \Gamma_r \to G$ is an admissible epimorphism. The* signature *of Δ is the vector $\mathbf{m} := (m_1, \ldots, m_r)$ where $m_i := \operatorname{ord}(\theta(\gamma_i))$. The* genus *of Δ is defined by the Riemann-Hurwitz formula:*

$$2(g(\Delta) - 1) = |G| \left(-2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right) \tag{2.1}$$

We let $\mathscr{D}^r$ or simply $\mathscr{D}$ denote the set of all r-data.

2.3. Orient S^2 by the outer normal. Consider smooth regular arcs $\tilde{\alpha}_i$ in S^2 joining p_0 to p_1 such that for $i \neq j$ $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ intersect only at p_0. Assume also that the tangent vectors at p_0 are all distinct and follow each other in counterclockwise order. Next consider loops α_i based at p_0 constructed as follows: α_i starts at p_0, travels along $\tilde{\alpha}_i$ until near p_i, there travels counterclockwise along a small circle around p_i, finally goes back to p_i again along $\tilde{\alpha}_i$. The circles have to be pairwise disjoint. We call the resulting set of generators $\{[\alpha_1], \ldots, [\alpha_r]\}$ a *geometric basis* of $\pi_1(S^2 - P, p_0)$. Once a geometric basis is fixed, there is a well-defined isomorphism

$$\chi : \Gamma_r \to \pi_1(S^2 - P, p_0)$$

such that $\chi(\gamma_i) = [\alpha_i]$.

2.4. The following geometric setting gives rise to data (and it is the main motivation for them). Let X be a compact (connected) Riemann surface. Assume that a finite group G acts effectively and holomorphically on X in such a way that $X/G = \mathbb{P}^1$. Let $P := \{p_1, \ldots, p_r\}$ be the critical values of $\pi : X \to \mathbb{P}^1 \cong S^2$. Fix $p_0 \in S^2 - P$ and a geometric basis $\{[\alpha_1], \ldots, [\alpha_r]\}$ with corresponding isomorphism $\chi : \Gamma_r \cong \pi_1(S^2 - P, p_0)$. Finally fix a point $\tilde{p}_0 \in \pi^{-1}(p_0)$. As is well-known there is a morphism $\tilde{\theta} : \pi_1(S^2 - P, p_0) \to G$ such that for $[\alpha] \in \pi_1(S^2 - P, p_0)$ the lifting of α starting at p_0 ends at $g \cdot p_0$ where $g = \bar{\theta}([\alpha])$. Since X is connected $\tilde{\theta}$ is surjective. Therefore $\Delta := (G, \theta := \tilde{\theta} \circ \chi)$ is an r-datum, $g(\Delta) = g(X)$ by the Riemann-Hurwitz formula and m_i is the cardinality of the stabilizer of points in $\pi^{-1}(p_i)$. We are going to show that each datum arises from a covering $X \to \mathbb{P}^1 = X/G$.

2.5. Assume from now on that $r \geq 3$ and denote by $\mathsf{T}_{0,r}$ the Teichmüller space in genus 0 and with r marked points. The definition of $\mathsf{T}_{0,r}$ is as follows. Fix $r+1$ distinct points $p_0, \ldots, p_r$ on S^2. For simplicity set $P = (p_1, \ldots, p_r)$. Consider triples of the form $(\mathbb{P}^1, x, [f])$ where $x = (x_1, \ldots, x_r)$ is an r-tuple of distinct points in $\mathbb{P}^1$ and $[f]$ is an isotopy class of orientation preserving homeomorphisms $f : (\mathbb{P}^1, x) \to (S^2, P)$. Two such triples $(\mathbb{P}^1, x, [f])$ and $(\mathbb{P}^1, x', [f'])$ are equivalent if there is a biholomorphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\varphi(x_i) = x'_i$ for any i and $[f] = [f' \circ \varphi]$. The Teichmüller space $\mathsf{T}_{0,r}$ is the set of all equivalence classes, see e.g. [1, Chap. 15] for more details.

2.6. Fix a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=}^r$ of $\pi_1(S^2 - P, p_0)$ with corresponding isomorphism $\chi : \Gamma_r \cong \pi_1(S^2 - P, p_0)$. Given an r-datum $\Delta = (G, \theta)$, the epimorphism $\theta \circ \chi^{-1}$ gives rise to a topological covering $\pi : \Sigma_0 \to S^2 - P$. By the topological part of Riemann's Existence Theorem this can be completed to a branched cover $\pi : \Sigma \to S^2$. Given a point $t = [\mathbb{P}^1, x, [f]] \in \mathsf{T}_{0,r}$, the homeomorphism f restricts to a homeomorphism of $\mathbb{P}^1 - x$ onto $S^2 - P$. We get an induced isomorphism $f_* : \pi_1(\mathbb{P}^1 - x, f^{-1}(p_0)) \cong \pi_1(S^2 - P, p_0)$. Thus $\theta \circ \chi^{-1} \circ f_* : \pi_1(\mathbb{P}^1 - x, f^{-1}(p_0)) \to G$ is an epimorphism and this gives rise to a topological covering $\pi_t^0 : C_t^0 \to \mathbb{P}^1 - x$. Here C_t^0 is an open differentiable surface. Since π_0 is a local diffeomorphism, there is a unique complex structure on C_t^0 making π_t^0 holomorphic. By the holomorphic part of Riemann's Existence Theorem C_t^0 and π_t^0 may be uniquely completed to a proper holomorphic map $\pi_t \colon C_t \to \mathbb{P}^1$ and the G-action extends to C_t. Moreover there is an isotopy class of homeomorphisms $\tilde{f}_t : C_t \to \Sigma$ that cover f_t. As t varies in $\mathsf{T}_{0,r}$ this construction yields a holomorphic map to the Teichmüller space of Σ

$$\Phi_\Delta : \mathsf{T}_{0,r} \longrightarrow \mathsf{T}_g \cong \mathsf{T}(\Sigma), \quad t \mapsto [C_t, [\tilde{f}_t]].$$

The group G embeds in the mapping class group of Σ, which we denote by Mod_g. This embedding depends on θ and we denote by $G_\theta \subset \mathrm{Mod}_g$ its image. The image of Φ_Δ coincides with $\mathsf{T}_g^{G_\theta}$, the set of fixed points of G_θ on T_g. As such it is a complex submanifold. We denote it by T_Δ.

The image of T_Δ in the moduli space M_g is an irreducible algebraic subvariety of dimension $(r-3)$ that we denote by M_Δ. (See e.g. [19, 6, 2, 7] for more details.) As explained in [19, p. 79] the map $\mathsf{T}_\Delta \to \mathsf{M}_\Delta$ factors through an intermediate variety $\tilde{\mathsf{M}}_\Delta$:

$$\mathsf{T}_\Delta \longrightarrow \tilde{\mathsf{M}}_\Delta \stackrel{\nu}{\longrightarrow} \mathsf{M}_\Delta.$$

The variety $\tilde{\mathsf{M}}_\Delta$ is the normalization of M_Δ. There is a finite cover $\mathsf{B}_\Delta \to \tilde{\mathsf{M}}_\Delta$ and a universal family

$$\pi_\Delta : \mathscr{C}_\Delta \to \mathsf{B}_\Delta.$$

We call it the *family of G-curves associated to* Δ. The proofs of these assertions can be found in [19] (where $T_g(H_0)$ corresponds in our notation to T_Δ, $\widetilde{\mathcal{M}}(H_0)$ to $\tilde{\mathsf{M}}_\Delta$, $\mathcal{M}(H_0)$ to M_Δ and $\tilde{\mathcal{M}}^{pure}(H_0)$ to B_Δ). Note that

$$\dim \mathsf{M}_\Delta = \dim \mathsf{B}_\Delta = r-3. \tag{2.2}$$

2.7. In this construction the choice of the base point p_0 is irrelevant. In fact (up to isomorphism) the ramified covering $\Sigma \to S^2$ only depends on $N := \ker\, \theta \circ \chi^{-1} \lhd \pi_1(S^2 - P, p_0)$. Two isomorphism $\pi(S^2 - P, p_0) \to \pi_1(S^2 - P, p_0')$ differ by an inner automorphism, so the map from normal subgroups of $\pi(S^2 - P, p_0)$ to those of $\pi(S^2 - P, p_0')$ is well defined. This proves that T_Δ and hence also M_Δ, $\tilde{\mathsf{M}}_\Delta$ and the family $\pi_\Delta : \mathscr{C}_\Delta \to \mathsf{B}_\Delta$ do not depend on the choice of the base point p_0.

2.8. On the other hand the construction of $\mathsf{T}_\Delta, \mathsf{M}_\Delta, \tilde{\mathsf{M}}_\Delta, \pi_\Delta$ does depend on the choice of the geometric basis. Let $\overline{\mathscr{B}} = \{[\bar{\alpha}_i]\}_{i=1}^r$ be another geometric basis. and let $\bar{\chi} : \Gamma_r \to \pi_1(S^2 - P, p_0)$ be the corresponding isomorphism. Then $\mu := \bar{\chi} \circ \chi^{-1} \in \mathrm{Aut}\, \pi(S^2 - P, p_0)$ has two special properties: 1) for every $i = 1, \dots, r$, $\mu([\alpha_i]) = [\bar{\alpha}_i]$ is conjugate to $[\alpha_j]$ for some j; 2) the induced homomorphism on the cohomology group $H_2(\pi_1(S^2 - P, p_0), \mathbb{Z})$ is the identity. By a variant of the Dehn-Nielsen Theorem (see e.g. [10, §8.2.7 p. 233] or [33, Thm. 5.7.1 p. 197]) there is an orientation-preserving diffeomorphism $\varphi : (S^2 - P, p_0) \to (S^2 - P, p_0)$ such that $\mu = \varphi_*$. Let Σ and $\bar{\Sigma}$ be the coverings of S^2 obtained from χ and $\bar{\chi}$. If $N = \ker\, \theta \circ \chi^{-1}$ and $\bar{N} = \ker\, \theta \circ (\bar{\chi})^{-1}$, then $\varphi_*(N) = \bar{N}$. By the Lifting Theorem there is an orientation-preserving diffeomorphism $\tilde{\varphi} : \Sigma \to \bar{\Sigma}$ that covers φ. This gives rise to a biholomorphism $\mathsf{T}(\Sigma) \to \mathsf{T}(\Sigma')$ which maps T_Δ constructed using χ to T_Δ constructed using $\bar{\chi}$. The identification $\mathsf{T}_g = \mathsf{T}(\Sigma)$ is defined up to the action of Mod_g and the discussion above shows that also T_Δ is well defined up to this action. In particular $\mathsf{M}_\Delta, \tilde{\mathsf{M}}_\Delta, \mathsf{B}_\Delta$ and π_Δ are completely independent of the choice of the geometric basis.

2.9. There is a representation

$$\rho : G \longrightarrow \mathrm{GL}\, H^0(C_t, K_{C_t}), \quad \rho(g) := (g^{-1})^*. \tag{2.3}$$

The equivalence class of this representation is independent of $t \in \mathsf{B}_\Delta$.

For later use we recall the following observation, already used in the proof of [14, Thm. 2.3].

Proposition 2.10. *Let G be a finite group of automorphisms of a curve C, and consider the subspace of invariants $H^0(C, 2K_C)^G$. Then the multiplication map*

$$m_C^G \colon S^2 H^0(C, K_C)^G \to H^0(C, 2K_C)^G$$

is surjective unless C is hyperelliptic (so of genus at least 2) and there is a small deformation C_t of the complex structure of C such that all elements of G remain holomorphic and the general curve C_t is not hyperelliptic.

In particular, for a fixed r-datum $\Delta = (G, \theta)$, the map m_C^G is surjective for the general $C \in \mathsf{B}_\Delta$.

Proof. Let g be the genus of C. The statement is obvious for $g \leq 1$ since the $G-$equivariant map $S^2(H^0(C, K_C)) \to H^0(C, 2K_C)$ is an isomorphism (among spaces of dimension g). If C is not hyperelliptic, then the statement follows similarly since the map $S^2(H^0(C, K_C)) \to H^0(C, 2K_C)$ is surjective by M. Noether's Theorem.

We can then assume that C is hyperelliptic. Let σ be the hyperelliptic involution. It is well-known that σ acts as the multiplication by -1 on $H^0(C, K_C)$, so trivially on $S^2(H^0(C, K_C))$, and that the multiplication map $S^2(H^0(C, K_C)) \to H^0(C, 2K_C)^{\langle\sigma\rangle}$ is surjective.

We distinguish two cases.

(1) If $\sigma \in G$ then the surjectivity of m_C^G follows by the surjectivity of the map $S^2(H^0(C, K_C)) \to H^0(C, 2K_C)^{\langle\sigma\rangle}$.
(2) If $\sigma \notin G$ we denote by $\tilde{G}$ the group of automorphisms of G generated by G and σ. Then $m_C^{\tilde{G}}$ is surjective. Moreover $S^2(H^0(C, K_C))^{\tilde{G}} = S^2(H^0(C, K_C))^G$ so we need $H^0(C, 2K_C)^G \cong H^0(C, 2K_C)^{\tilde{G}}$, that is equivalent to $H^0(C, 2K_C)^G \subset H^0(C, 2K_C)^{\langle\sigma\rangle}$. Dualizing, this is equivalent to $H^1(C, T_C)^G \subset H^1(C, T_C)^{\langle\sigma\rangle}$, which amounts to asking that every small deformation of the pair (C, G) remain hyperelliptic.

□

2.11. We notice that the exceptional case in Proposition 2.10 occurs. Consider for example family (27) in [12, Table 2]. A direct computation shows that this 3-dimensional family of curves of genus 3 with an action of $(\mathbb{Z}/2\mathbb{Z})^2$ intersects the hyperelliptic locus in the 2-dimensional family of curves with an action of $(\mathbb{Z}/2\mathbb{Z})^3$ considered in [29, Table 2 - Five critical values - (b)].

If C belongs to this latter family, then $3 = h^0(C, 2K_C)^G \neq h^0(C, 2K_C)^{\tilde{G}} = 2$ and therefore m_C^G has corank 1.

2.12. Consider now a datum Δ and the family $\pi_\Delta : \mathscr{C}_\Delta \to \mathsf{B}_\Delta$. As t varies in B_Δ, the domain and codomain of $m_{C_t}^G$ do not change in dimension. Set

$$N(\Delta) := \dim \left(S^2 H^0(C_t, K_{C_t})\right)^G \tag{2.4}$$

Theorem 2.13. *If $g = g(\Delta) \geq 2$ and*

$$N(\Delta) = r - 3, \tag{$*$}$$

then $\overline{j(\mathsf{M}_\Delta)}$ (closure in A_g) is a special subvariety of PEL type of A_g that is generically contained in the Jacobian locus.

(See [12, Thm. 3.9] and [13, Thm. 3.7].)

2.14. The idea of Theorem 2.13 is that from Δ one can construct both M_Δ and a Shimura subvariety $\mathsf{Z}_\Delta \subset \mathsf{A}_g$ with $N(\Delta) = \dim \mathsf{Z}_\Delta$. By construction $j(\mathsf{M}_\Delta) \subset \mathsf{Z}_\Delta$ and both M_Δ and Z_Δ are irreducible algebraic subvarieties. By (2.2) $\dim \mathsf{M}_\Delta = r - 3$. Since j is an injective morphism of algebraic varieties, when $g \geq 2$ we always have $N \geq r - 3$. If $(*)$ holds, then $j(\mathsf{M}_\Delta)$ is dense in Z_Δ.

2.15. Note also that (when $g \geq 2$) for any $t \in \mathsf{B}_\Delta$ we have $\dim H^0(2K_{C_t})^G = \dim H^1(T_{C_t})^G = \dim \mathsf{B}_\Delta = r - 3$. Hence condition $(*)$ in Theorem 2.13 coincides with condition $(*)$ of the Introduction. It amounts to asking that domain and codomain of $m_{C_t}^G$ have the same dimension. By Proposition 2.10 this is then equivalent to asking that, for general t, $m_{C_t}^G$ is injective.

2.16. We now wish to prove a lemma that is helpful to rule out *a priori* some groups.

Let $\Delta = (G, \theta)$ be a datum and let H be a normal subgroup of G. Set $K := G/H$ and let $\pi : G \to K$ be the canonical projection. The composition $\pi \circ \theta : \Gamma_r \to G \to K$ is an epimorphism, but it is not necessary admissible, since some of the $\gamma_i \in \Gamma_r$ might map to 1. We can throw them away obtaining an admissible epimorphism $\bar{\theta} : \Gamma_s \to K$ for some $s \leq r$. In terms of spherical generators this means the following: if $\theta(\gamma_i) = g_i$ and $k_i = \pi(g_i)$, then $\bar{\theta} = (k_1, \ldots, k_r)$ where we omit all the k_i that equal 1. So we get a new datum $\bar{\Delta} = (K, \bar{\theta})$. This corresponds to the following geometric situation. Δ gives rise to the family $\pi_\Delta : \mathscr{C}_\Delta \to \mathsf{B}_\Delta$. We can quotient each fibre C_t by H getting a curve $F_t := C_t/H$ on which K acts:

$$\begin{array}{ccc} C_t & \xrightarrow{\quad p \quad} & F_t = C_t/H \\ & {\scriptstyle \pi} \searrow \quad \swarrow & \\ & \mathbb{P}^1 = C_t/G = F_t/K. & \end{array}$$

The curves F_t form a family $\mathscr{F} \to \mathsf{B}_\Delta$. If $g(F_t) \geq 2$, out of the datum $\bar{\Delta}$ we can form the family $\mathscr{C}_{\bar{\Delta}} \to \mathsf{B}_{\bar{\Delta}}$ as explained in 2.6. Then $\mathscr{F}$ is a pull-back of this family, i.e. $f^*\mathscr{C}_{\bar{\Delta}} = \mathscr{F}$ for some holomorphic map $f : \mathsf{B}_\Delta \to \mathsf{B}_{\bar{\Delta}}$.

Lemma 2.17. *In the above situation, assume that $g(F) \geq 2$. If $(*)$ holds for Δ, then it holds also for $\bar{\Delta}$.*

Proof. Write for simplicity $C = C_t$ and $F = F_t$. We have two pull-back maps:

$$p^* : H^0(K_F) \hookrightarrow H^0(K_C), \quad p^* : H^0(2K_F) \hookrightarrow H^0(2K_C).$$

From the first one we obtain also an injection

$$f := S^2 p^* : S^2 H^0(K_F) \hookrightarrow S^2 H^0(K_C).$$

Since $p^* H^0(K_F) = H^0(K_C)^H$ then

$$f((S^2 H^0(K_F))^K) \subset (S^2 H^0(K_C))^G.$$

Thus, we get a commutative diagram

$$\begin{array}{ccc} \big(S^2 H^0(K_F)\big)^K & \xrightarrow{m_F^K} & H^0(2K_F)^K \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p^*} \\ \big(S^2 H^0(K_C)\big)^G & \xrightarrow{m_C^G} & H^0(2K_C)^G. \end{array}$$

from which

$$m_C^G \text{ injective} \Rightarrow m_F^K \text{ injective}.$$

As explained in 2.15, if $(*)$ holds for Δ, then m_C^G is injective for general C and therefore m_F^K is injective for general F, so $N(\bar{\Delta}) = S^2 H^0(K_F)^K \leq H^0(2K_F)^K$. But since $g(F) \geq 2$, the discussion in 2.14 shows that $N(\bar{\Delta} \geq s - 3 = H^0(2K_F)^K$. Thus $N(\bar{\Delta}) = s - 3$, i.e. $\bar{\Delta}$ satisfies $(*)$. □

3. Avoiding unnecessary computations

This section collects several results that allow to rule out *a priori* various cases avoiding some parts, sometimes really substantial, of the computation. We briefly explain its contents.

Lemmata 3.1 and 3.2 use the same ideas underlying the proof of the Hurwitz theorem to ensure that signatures exist only in some ranges. Theorem 3.3) summarizes results of Moonen and Mohajer-Zuo, saying that no new counter-examples exist in certain cases.

In 3.4 we introduce spherical systems of generators, recall the Chevalley-Weil formula, define refined passports and show that $N(\Delta)$ only depends on the refined passport of the generators. We then recall Eichler's formula. It is used in the proof of Theorem 3.12, which says that no counter-example exists with $G = (\mathbb{Z}/2\mathbb{Z})^k$ for $g \geq 4$. Its Corollary 3.13 is the main tool to cut down the number of computations to be done. Other such tools are Frobenius' test (Corollary 3.15) and an elementary observation on the abelianization of a group admitting a spherical system of generators (§3.16).

Lemma 3.1. *If (G, θ) is an r-datum of genus g and G contains an element of order $> 4(g-1)$, then either $r = 3$, i.e. the family is 0-dimensional, or it coincides with family (5) in* [12, Table 2].

If $x \in G$ has order $> 4(g-1)$, then by definition $H := \langle x \rangle$ is a *large automorphism group* of C. So the Lemma follows immediately from Proposition 4.5 in [21]. The idea of using upper bounds for the order of single elements of G comes from Corollary 5.10 in [4], where the classical bound of Wiman was used. The theorem of Kulkarni that we use here is more precise.

Lemma 3.2. *Let $\Delta = (G, \theta)$ be an r-datum with genus $g \geq 2$ and $r \geq 4$. If the datum corresponds to an action of G on a smooth curve X with $X/G = \mathbb{P}^1$, then (a) $r \leq 2g+2$ with equality only for X hyperelliptic and G generated by the hyperelliptic involution, (b) $r \leq 4 + \frac{4(g-1)}{d}$ and (c) $|G| \leq 12(g-1)$.*

Proof. The arguments are extremely classical, but for the reader's convenience we give the proof. Set $d := |G|$, $\delta := \sum_{i=1}^r \frac{1}{m_i}$ and $\mu := r - 2 - \delta$. By the Riemann-Hurwitz formula,

$$2(g-1) = d \cdot \mu \tag{3.1}$$

Assume $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$. Since $g \geq 2$, $\mu > 0$. For $x > 0$ set $f(x) := 1 - 1/x$. Then $\mu = \sum_{i-1}^r f(m_i) - 2$. Since f is increasing $\mu \geq r \cdot f(2) - 2 = (r-4)/2$. Using $d \geq 2$ and (3.1), this gives $g-1 \geq (r-4)/2$, i.e. the inequality in (a). If equality holds $|G| = 2$, so the curves are hyperelliptic. By a dimensional count the family coincides with that of hyperelliptic curves. This proves (a).

Set $A(r) := \{x = (x_1, \ldots, x_r) \in \mathbb{Z}^r : x_i \geq 2, \sum_{i-1}^r f(x_i) > 2\}$ and

$$\bar{\mu}(r) := \min_{x \in A(r)} \left\{ \sum_{i=1}^r f(x_i) - 2 \right\}$$

Using the fact that f is strictly increasing one verifies that for $r = 4$ the minimum is achieved at $x = (2,2,2,3)$ and $\bar{\mu}(4) = 1/6$, while for $r \geq 5$ the minimum is achieved at $x = \underbrace{(2, \ldots, 2)}_{r \text{ times}}$ and $\bar{\mu}(r) = r/2 - 2$. So for any $r \geq 4$ we have $\bar{\mu}(r) \geq (r-4)/2$. Let now $\mathbf{m}$ be the signature of the datum (G, θ). Then $\mathbf{m} \in A(r)$, so $\mu \geq \bar{\mu}(r)$. Thus (3.1) gives $2(g-1)/d \geq \bar{\mu}(r) \geq (r-4)/2$, which is the inequality in (b).

If $r = 4$, (3.1) gives $2(g-1)/d \geq \bar{\mu}(4) = 1/6$, which is equivalent to the inequality in (c). If $r > 4$ in the same way we get $d \leq 4(g-1)/(r-4) \leq 4(g-1)$. But $4(g-1)/(r-4) \leq 4(g-1) \leq 12(g-1)$. Hence the inequality in (c) holds for every value of r. □

Theorem 3.3. *The data $\Delta = (G, \theta)$ satisfying $(*)$ with G cyclic or with G abelian and $r = 4$ are Hurwitz equivalent to those mentioned in Theorem 1.5. Moreover for such data $(*)$ is necessary for Z_Δ to be a Shimura subvariety.*

These results are due to Moonen [25] and Mohajer-Zuo [24, Thms. 3.1 and 6.2].

3.4. If G is a finite group, giving an r-datum $\Delta = (G, \theta)$ is equivalent to giving a list of generators $g_1, \ldots, g_r$ of G such that $g_i \neq 1$ for any i and subject to the constraint $g_1 \cdots g_r = 1$. Indeed, this defines an epimorphism $\theta \colon \Gamma_r \to G$ by $\theta(\gamma_i) = g_i$. From now on we will write $\Delta \in \mathscr{D}^r$ as $\Delta = (G, g_1, \ldots, g_r)$, and we will call $(g_1, \ldots, g_r)$ a *spherical system of generators* of the group G.

Let χ_ρ denote the character of the representation ρ defined in (2.3). As explained in [12, §§2.9ff] the number $N(\Delta)$ in (2.4) can be computed from χ_ρ:

$$N(\Delta) = \frac{1}{2|G|} \sum_{a \in G} \big(\chi_\rho(a^2) + \chi_\rho(a)^2\big). \tag{3.2}$$

So to test (∗) one needs to compute χ_ρ for a datum Δ. There are two ways to do that: using Eichler's trace formula or the Chevalley-Weil formula. We need both and we start from the Chevalley-Weil formula.

3.5. Next, fix a datum $\Delta = (G, g_1, \dots, g_r)$ and let $m_j := \operatorname{ord}(g_j)$ as usual. Denote by $\operatorname{Irr} G$ the set of irreducible characters of G. For each $\chi \in \operatorname{Irr} G$ fix a representation σ_χ with character χ. For $n \in \mathbb{N}$, $n > 0$ set $\zeta_n := \exp(2\pi i/n)$. If $\chi \in \operatorname{Irr} G$, $1 \le j \le r$ and $0 \le \alpha < m_j$, denote by $N_{j,\alpha}$ the multiplicity of $\zeta_{m_j}^\alpha$ as an eigenvalue of $\sigma_\chi(g_j)$.

Theorem 3.6 (Chevalley–Weil)**.** *If $\Delta = (G, g_1, \dots, g_r)$ is a datum for the Galois covering $C \to \mathbb{P}^1$, then the multiplicity μ_χ of $\chi \in \operatorname{Irr} G$ in ρ is*

$$\mu_\chi = -\deg \chi + \sum_{j=1}^{r} \sum_{\alpha=0}^{m_j-1} N_{j,\alpha} \frac{\alpha}{m_j} + \varepsilon, \tag{3.3}$$

where $\varepsilon = 1$ if χ is the trivial character and $\varepsilon = 0$ otherwise.

A nice reference for the Chevalley-Weil formula is [18, Ch. 1]. Our implementation uses this formula to compute χ_ρ and hence $N(\Delta)$. In fact we use the same algorithm as Gleißner, which is based in turn on [31], but with code optimized for our setting (see 4.6).

Definition 3.7. *Given a finite group G let $\mathcal{C}_G$ or simply $\mathcal{C}$ denote the set of conjugacy classes of G. The symmetric group Σ_r acts on $\mathcal{C}_G^r$. A* refined passport *with r branch points for the group G is an element of $\mathcal{C}_G^r/\Sigma_r$. Thus a refined passport is an undordered sequence of conjugacy classes of G. Given $\Delta = (G, g_1, \dots, g_r)$, the* refined passport *of Δ is the class of $([g_1], \dots, [g_r])$ in $\mathcal{C}_G^r/\Sigma_r$.*

Note that this definition is slightly different from those of [22] and [28]: we do not assume that a refined passport comes from a datum.

3.8. It is clear that the numbers $N_{j,\alpha}$ defined in 3.5 do not change if g_j is replaced by another element $g'_j \in G$ which is conjugate to g_j. Another observation is that obviously the sum in (3.3) is independent of the order. Thus $N(\Delta)$ depends only on the refined passport of Δ. This elementary observation is at the basis of our approach to the computation.

Lemma 3.9. *Let G be a finite group and let $C_i \in \mathcal{C}_G$ for $i = 1, \dots, r$. Assume that there is a datum $\Delta = (G, g_1, \dots, g_r)$ with $g_i \in C_i$ for $i = 1, \dots, r$. Then for any $\sigma \in \Sigma_r$ there is a datum $(G, \gamma_1, \dots, \gamma_r)$ such that $\gamma_i \in C_{\sigma_i}$ for $i = 1, \dots, r$.*

Proof. Since Σ_r is generated by simple transpositions, it is enough to prove the result for $\sigma = (j, j+1)$, $1 \leq j < r$. Set

$$\gamma_i = g_i, \text{ for } i \notin \{j, j+1\}, \qquad \gamma_j = g_j g_{j+1} g_j^{-1}, \qquad \gamma_{j+1} = g_j.$$

Then $(G, \gamma_1, \ldots, \gamma_r)$ is still a datum and $\gamma_i \in C_{\sigma_i}$ for any i. □

3.10. Now we turn to Eichler's formula, which is important to rule out a class of groups. Recall that if $a \in G$, $p \in C$ and $a \cdot p = p$, then $da(p) \in \operatorname{End} T_p C$ is multiplication by a root of unity, which we denote simply by $da(p)$.

Theorem 3.11 (Eichler Trace Formula)**.** *If $a \in G$, $a \neq 1$ then*

$$\chi_\rho(a) = 1 - \sum_{p \in \operatorname{Fix}(a)} \frac{1}{1 - da(p)}. \tag{3.4}$$

See e.g. [11, Thm. V.2.9, p. 264].

Theorem 3.12. *Let $\Delta = (G, g_1, \ldots, g_r)$ be a datum corresponding to a covering $C \to \mathbb{P}^1$ with $G \cong (\mathbb{Z}/2\mathbb{Z})^k$. If $g(C) \geq 4$, then $(*)$ does not hold for Δ.*

Proof. The families fulfilling condition $(*)$ with genus up to 7 have been classified in [12, Theorems 5.4 and 5.5] and are all listed in [12, Table 2]: inspecting the table we see that we may assume $g(C) \geq 8$.

Since all elements a in G, $a \neq 1$, have order 2, by the Hurwitz formula

$$\chi_\rho(1) = g(C) = 1 + \frac{|G|}{4}(r-4) = 1 + 2^{k-2}(r-4).$$

Moreover for all $p \in \operatorname{Fix}(a)$, $da(p) = -1 \in \mathbb{R}$ and then, by (3.4) for all $a \in G$, $\chi_\rho(a) \in \mathbb{R}$. In particular all summands in the expression of N in (3.2) are real numbers and

$$\begin{aligned} N(\Delta) = \frac{1}{2|G|} \sum_{a \in G} \big(\chi_\rho(a^2) + \chi_\rho(a)^2\big) = \frac{1}{2^{k+1}} \sum_{a \in G} \big(\chi_\rho(1) + \chi_\rho(a)^2\big) \geq \\ \geq \frac{\big(\sum_{a \in G} \chi_\rho(1)\big) + \chi_\rho(1)^2}{2^{k+1}} = g(C)\left(\frac{1}{2} + \frac{g(C)}{2^{k+1}}\right) = \\ = g(C)\left(\frac{1}{2} + \frac{1}{2^{k+1}} + \frac{r-4}{8}\right) = g(C)\left(\frac{1}{2^{k+1}} + \frac{r}{8}\right) > g(C)\left(\frac{r}{8}\right) \geq r \end{aligned}$$

contradicting $(*)$. □

Considering Lemma 2.17 we deduce the following stronger result:

Corollary 3.13. *Let $\Delta = (G, g_1, \ldots, g_r)$ be a datum corresponding to a covering $C \to \mathbb{P}^1$. If there is a surjective map $G \to (\mathbb{Z}/2\mathbb{Z})^4$, then $(*)$ does not hold for Δ.*

Proof. Assume by contradiction that $(*)$ holds.

Let H be the kernel of the surjection $G \to (\mathbb{Z}/2\mathbb{Z})^4$ and consider the family of the curves $F_t = C_t/H \to \mathbb{P}^1$ as in 2.16. They are Galois covers with datum $\bar{\Delta} = ((\mathbb{Z}/2\mathbb{Z})^4, h_1, \cdots, h_s)$.

Since each set of generators of $(\mathbb{Z}/2\mathbb{Z})^4$ has cardinality at least 4, then $s \geq 5$. This implies $g(F) \geq 2$ by the Hurwitz formula and $g(F) \leq 4$ by Lemma 2.17 and Theorem 3.12.

The Galois covers of $\mathbb{P}^1$ with genus among 2 and 4 having 4 or more branch points are listed in [12, Table 2]: we see that the group $(\mathbb{Z}/2\mathbb{Z})^4$ does not occur, reaching an absurd. □

The Galois group G of family (34) in [12, Table 2] admits $(\mathbb{Z}/2\mathbb{Z})^3$ as a quotient. Thus one cannot improve the above Corollary by substituting $(\mathbb{Z}/2\mathbb{Z})^4$ with one of its proper quotients. In fact applying Lemma 2.17 to this case yields one of the families of elliptic curves mentioned after Theorem 1.5.

There is another useful criterion, already used by Breuer [5] and Paulhus [28]. Indeed, for some elements c, one can ascertain *a priori* that $\pi^{-1}(c) = p^{-1}(\tilde{c})$ does not contain any system of generators at all. This is based on a theorem of Frobenius. (See [22, p. 406] for a proof.)

Theorem 3.14 (Frobenius' formula)**.** *Given a finite group G and conjugacy classes $C_1, \dots, C_r$, the number of r-ples $(g_1, \dots, g_r) \in C_1 \times \cdots \times C_r$ such that $\prod g_i = 1$ is*

$$\frac{|C_1| \cdots |C_r|}{|G|} \sum_{\chi \in \operatorname{Irr} G} \frac{\chi(C_1) \cdots \chi(C_r)}{\chi(1)^{r-2}}.$$

Notice that this condition is independent of the order.

Corollary 3.15. *Let G be a group and $(C_1 \dots, C_r)$ a refined passport. If*

$$\sum_{\chi \in \operatorname{Irr} G} \frac{\chi(C_1) \cdots \chi(C_r)}{\chi(1)^{r-2}} = 0,$$

then there is no datum $(G, g_1, \dots, g_r)$ with refined passport $(C_1, \dots, C_r)$.

3.16. We conclude with a useful elementary observation. Assume that a group G admits a system of spherical generators $(g_1, \dots, g_r)$ with signature $(m_1, \dots, m_r)$. Decompose its abelianization $\operatorname{Ab} G = \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_p\mathbb{Z}$ with $k_1 | \cdots | k_p$ (i.e. the k_i's are the invariant factors). Since for any j, $\operatorname{Ab} G$ is generated by the images of $g_1, \dots, \hat{g}_j, \dots, g_r$, it follows that $p \leq r - 1$ and that k_p divides $\operatorname{lcm}(m_1, \dots, \hat{m}_j, \dots, m_r)$ for any j.

4. The algorithm

4.1. Given a group G, let $\mathcal{C}_G$ be the set of its conjugacy classes. Recall from Definition 3.7 that a refined passport on G with r branch points is an unordered sequence of r conjugacy classes of G, i.e. an element of $\mathcal{C}_G^r/\Sigma_r$. If a refined passport contains a spherical system of generators $\Delta = (g_1, \dots, g_r)$, $g(\Delta)$ and $N(\Delta)$ only depend on the refined passport of Δ. We will say that a refined passport is a *counter-example of genus g* if it contains a spherical system of generators Δ with $g(\Delta) = g$ such that $(*)$ holds. Notice that refined passports that satisfy $(*)$ formally but do not contain a spherical system of generators are excluded by this definition. The group $\operatorname{Aut} G$ acts both on $\mathcal{C}_G$ and on the set of refined passports.

4.2. We illustrate an algorithm to attack the following:

Problem 4.1. *For fixed $g \geq 2$, list groups G and counter-examples of genus g on G with $r \geq 4$ branch points, one for each orbit of* $\mathrm{Aut}(G)$*, leaving aside those with G cyclic and those with G abelian and $r = 4$.*

Our basic strategy is to fix r, and then choose one refined passport of genus g with r branch points in each $\mathrm{Aut}(G)$-orbit. If $(*)$ holds, it then suffices to determine whether the refined passport contains a system of spherical systems of generators.

4.3. As in [2, 12], we use signature as an invariant. Using the notation of Definition 2.2 signature defines a map

$$\mathscr{D}^r \to \mathbb{N}^r, \quad (g_1, \dots, g_r) \mapsto (\mathrm{ord}(g_1), \dots, \mathrm{ord}(g_r)).$$

Since the order of an element only depends on its conjugacy class, the signature of a spherical system of generators $(g_1, \dots, g_r)$ only depends on the conjugacy classes $([g_1], \dots, [g_r])$. Corresponding to the fact that refined passports are taken up to reordering (Lemma 3.9), signatures can be considered up to permutation, i.e. we can restrict to signatures satisfying $m_1 \leq \dots \leq m_r$.

We iterate over the order $d = |G|$. For fixed d, let $\mathfrak{S}_{d,g}$ be the set of finite sequences $\mathbf{m} = (m_1, \dots, m_r)$ such that

(S1) $4 \leq r \leq \frac{4(g-1)}{d} + 4$ and $d \leq 12(g-1)$;
(S2) each m_i is a divisor of d;
(S3) $1 < m_i < d$;
(S4) g and $\mathbf{m}$ satisfy (2.1);
(S5) $m_1 \leq \dots \leq m_r$;

By Lemma 3.2, the signature of a spherical system of generators Δ with $r \geq 4$ and $g(\Delta) = g$ must satisfy (S1); the restriction $r \geq 4$ ensures that the family is positive-dimensional, see (2.2); the restriction $m_i < d$ in (S3) is motivated by the fact that we are only interested in noncyclic groups G.

The set of "admissible" signatures $\mathfrak{S}_{d,g}$ is computed by Algorithm 1. In the implementation, we found it convenient to compute each $\mathfrak{S}_{d,g}$ for $2 \leq g \leq g_{max}$ simultaneously, and then store the result on disk for later retrieval, rather than iterate over g; this prevents repeating some computations.

4.4. Elements of $\mathcal{C}^r_G/\Sigma_r$ (i.e. refined passports) can be viewed as *multisets*. Given a set X, a multiset of elements of X can be defined as a set $\{(x_1, n_1), \dots, (x_k, n_k)\}$ where the x_i are pairwise disjoint elements of X and the n_i are nonnegative integers representing the *multiplicity* of x_i. In fact, it is customary to require the n_i to be positive, but it will be convenient for our purposes to allow them to be zero as well. We will write a multiset as $\{x_1^{n_1}, \dots, x_k^{n_k}\}$. A set $\{x_1, \dots, x_k\}$ can be identified with the multiset $\{x_1^1, \dots, x_k^1\}$, and the union of two multisets is defined in the obvious way by adding multiplicities.

It will also be convenient to represent elements of $\mathfrak{S}_{d,g}$ as multisets of integers $\{m_1^{n_1}, \dots, m_k^{n_k}\}$; for instance, the signature $(2, 2, 3, 3, 3)$ will be represented by the multiset $\{2^2, 3^3\}$.

4.5. Problem 4.1 can then be addressed by iterating through the signatures $\mathbf{m} \in \mathfrak{S}_{d,g}$ computed in Algorithm 1 and groups G of order d. A refined passport with signature $\mathbf{m}$ only exists on a group G if there is at least one element of order m_j for every $m_j \in \mathbf{m}$; we therefore discard groups and signatures that do not satisfy this condition. More groups and signatures can be eliminated by taking advantage of Lemma 3.1, Corollary 3.13 and the observation in 3.16. This procedure is displayed in Algorithm 2, which reduces the problem to identifying counter-examples for fixed group and signature. Notice that on line 25 the signature $\{m_1^{n_1}, \dots, m_k^{n_k}\}$ is converted into a multiset

$$\{A_1^{n_1}, \dots, A_k^{n_k}\} \subset \mathcal{P}(\mathcal{C}_G), \tag{4.1}$$

where each A_i is the subset of $\mathcal{C}_G$ of conjugacy classes of order m_i. This is the basis for the recursion of Algorithm 4.

4.6. At this point we need to determine the counter-examples with a given signature $\mathbf{m}$ and group G. This is achieved by picking one refined passport with signature $\mathbf{m}$ in each $\mathrm{Aut}(G)$-orbit, then verifying whether $(*)$ holds and the refined passport contains a spherical system of generators.

The iteration through one refined passport in each $\mathrm{Aut}(G)$-orbit is performed in Algorithm 4. A refined passport with signature $\{m_1^{n_1}, \dots, m_k^{n_k}\}$ is obtained by choosing n_i conjugacy classes with order m_i for each $1 \leq i \leq k$; in terms of (4.1), for each i we must choose a multiset S_i of n_i elements of A_i, counted with multiplicities. We can write S_i in a unique way as a union of sets $\bigcup_j B_{ij}$, where $B_{i1} \supset B_{i2} \supset \dots$ is a definitely empty sequence of subsets of A_i; this means that the multiplicity of C in S_i is the number of indices j such that C is in B_{ij}. Thus, iterating through the possible multisets S_i is equivalent to iterating through sequences

$$A_i \supset B_{i1} \supset B_{i2} \supset \dots, \quad \sum |B_{ij}| = n_i.$$

This must be repeated for each $i = 1, \dots, k$.

Our goal is to perform a similar iteration by choosing a single element in each $\mathrm{Aut}(G)$-orbit. To begin with, our algorithm picks a subset B of A_k with $1 \leq h \leq n_k$ elements, representing B_{k1} in the notation above. For each choice of B, the function recursively iterates through refined passports obtained by taking the union of B and a refined passport with n_i elements in each A_i, $i < k$ and $n_k - h$ elements in B. The recursive call iterates through one refined passport for each H-orbit, where H is the stabilizer of B in $\mathrm{Aut}(G)$. Top-level iteration over one subset B for each $\mathrm{Aut}(G)$-orbit completes the algorithm.

This approach requires a much lower amount of memory than determining all possible refined passports first and then picking one in each $\mathrm{Aut}(G)$-orbit. Notice also that the refined passports produced by the algorithm are elaborated sequentially, and not stored simultaneously into memory. Nevertheless, the algorithm must iterate through one subset of A_k for each $\mathrm{Aut}(G)$-orbit, and we are not aware of any efficient way of doing this without storing all subsets of fixed cardinality in memory. This is the one point in the whole algorithm where memory consumption can be significant.

Algorithm 3 determines whether a refined passport is a counter-example; first, the condition of Theorem 3.14 is verified, i.e. whether $\sum_\chi \frac{\chi(C_1)\cdots\chi(C_r)}{\chi(1)^{r-2}}$ is nonzero; if so, we will say that $(C_1, \ldots, C_r)$ passes Frobenius' test. Then, condition $(*)$ is tested by selecting random elements inside each C_i and computing $N(\Delta)$ by (3.3). Notice that each term $\sum_\alpha N_{j,\alpha}\alpha/m_j$ appearing in (3.3) only depends on the corresponding g_j, and the characters χ only depend on the group G. Thus, it suffices to compute these data at the beginning of the computation, when G is fixed, making the computation of (3.3) in the iteration quite fast. Only when both Frobenius' test and $(*)$ hold does the algorithm perform the most computationally expensive step, namely checking whether $C_1 \times \cdots \times C_r$ contain a spherical system of generators, by straightforward iteration.

4.7. For abelian groups G, conjugacy classes contain a single element, and the algorithm can be improved.

First, observe that Frobenius' test is useless in this case: the product $C_1 \times \cdots \times C_r$ contains a single element $(g_1, \ldots, g_r)$, so the condition $\prod g_i = 1$ is best verified directly.

Second, since refined passports contain a single element of G^r, we effectively iterate through elements of G^r. However, in a spherical system of generators $(g_1, \ldots, g_r)$ any element is determined by the others, so we can iterate through "short sequences" $(g_1, \ldots, \hat{g}_j, \ldots, g_r)$. Thus, we proceed as follows.

We fix m_j in $\mathbf{m} = (m_1, \ldots, m_r)$ such that the number of elements of G with order m_j is largest; then, we use a scheme analogous to Algorithm 4 to iterate through $(r-1)$-ples $(g_1, \ldots, \hat{g}_j, \ldots, g_r) \in G^{r-1}$ with signature $(m_1, \ldots, \hat{m}_j, \ldots, m_r)$, one for each $\mathrm{Aut}(G)$-orbit. We then define g_j as the inverse of $g_1 \cdots \hat{g}_j \cdots g_r$; if g_j has order m_j, the $(g_1, \ldots, g_r)$ is a candidate for a spherical system of generators with signature $\mathbf{m}$. At this point, we test condition $(*)$ and, if it holds, whether the elements $g_1, \ldots, g_r$ generate the group G.

Algorithm 1: Computing the signatures

input : integers $g \geq 2, d \geq 2$
output: the set of signatures $\mathfrak{S}_{d,g}$

Function $\mathfrak{S}_{d,g}(d, g)$
    if *d prime* **then**
        return $\emptyset$ `// (S3) cannot be satisfied`
    $\mathfrak{S}_{d,g} \leftarrow \emptyset$
    for *r satisfying (S1)* **do**
        $D \leftarrow \{n \in \mathbb{N} \mid 2 \leq n < d,\ n \text{ divides } d\}$;
        for $m_1, \ldots, m_r \in D$, $m_1 \leq \cdots \leq m_r$ **do**
            if *$(m_1, \ldots, m_r)$ satisfies (S4) and (S5)* **then**
                insert $(m_1, \ldots, m_r)$ in $\mathfrak{S}_{d,g}$
    return $\mathfrak{S}_{d,g}$

Algorithm 2: Find counter-examples of genus g with $r \geq 4$ branch points

input : an integer $g \geq 2$

output: counter-examples of genus g with $r \geq 4$ branch points, one for each $\mathrm{Aut}(G)$-orbit

Function *admissible(*G,$\mathbf{m}$*)*
- $\mathcal{O} \leftarrow \{\mathrm{ord}(g) \mid g \in G\}$;
- **if** $r = 4$ *and* G *abelian* **then**
  - **return** false;
- **else if** $r > 4$ *and* G *cyclic* **then**
  - **return** false;
- **else if** *some* m_i *is not in* $\mathcal{O}$ **then**
  - **return** false
- **else if** $g > 2$ *and some* $o \in \mathcal{O}$ *is greater than* $4(g-1)$ **then**
  - **return** false `// Lemma 3.1`
- **else**
  - decompose the abelianization of G as $\mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_p\mathbb{Z}$, with each k_i dividing k_{i+1};
  - **if** *at least* 4 *elements in* $(k_1, \dots, k_p)$ *are even* **then**
    - **return** false `// G surjects over` $(\mathbb{Z}/2\mathbb{Z})^4$ `(Corollary 3.13)`
  - **if** $p \geq r$ **then**
    - **return** false `// r − 1 elements cannot generate G`
  - **else if** *exists* j *such that* $k_p \nmid \mathrm{lcm}(m_1, \dots, \hat{m}_j, \dots, m_r)$ **then**
    - **return** false `// §3.16`
  - **return** true `// passed all tests`

for $2 \leq d \leq 12(g-1)$ **do**
- determine $\mathfrak{S}_{d,g}$ by Algorithm 1;
- **for** $\mathbf{m} = \{m_1^{k_1}, \dots, m_k^{r_k}\}$ *in* $\mathfrak{S}_{d,g}$ **do**
  - **for** G *group of order* d **do**
    - **if** *admissible(*G,$\mathbf{m}$*)* **then**
      - **for** $1 \leq i \leq k$ **do**
        - $A_i \leftarrow \{C \in \mathcal{C}_G \mid \mathrm{ord}(C) = m_i\}$
      - CounterExamplesIn $(G, \{A_1^{r_1}, \dots, A_k^{r_k}\})$ `// Find counter-examples for group G and signature m using Algorithm 4`

Algorithm 3: Determine whether a refined passport is a counter-example

input : a group G, a refined passport $(C_1, \ldots, C_r)$
output: true if the refined passport is a counter-example, false otherwise

Function *IsCounterExample(G, $C_1, \ldots, C_r$)*
  if $(C_1, \ldots, C_r)$ *passes Frobenius' test and* $N = r - 3$ **then**
    for $(g_1, \ldots, g_{r-1})$ *in* $C_1 \times \cdots \times C_{r-1}$ **do**
      $g_r \leftarrow (g_1 \cdots g_{r-1})^{-1}$;
      if $g_r \in C_r$ *and* $\langle g_1, \ldots, g_{r-1} \rangle = G$ **then**
        return true
    return false

Algorithm 4: Find counter-examples for fixed group and signature

input : A group G and a nonempty multiset $\{A_1^{n_1}, \ldots, A_k^{n_k}\}$ where each A_i is a nonempty set of conjugacy classes of G and each n_i is a nonnegative integer
output: Counter-examples obtained by choosing n_i elements in each A_i, one for each $\mathrm{Aut}(G)$-orbit

Function *CounterExamplesIn(G, $\{A_1^{n_1}, \ldots, A_k^{n_k}\}$, $S = \emptyset$, $H = \mathrm{Aut}(G)$)*
  `// This is a recursive function using two arguments with default values:` S `is a multiset of conjugacy classes of` G`;` H `is a subgroup of` $\mathrm{Aut}(G)$ `acting on each` A_i
  if $k = 0$ **then**
    if *IsCounterExample(G,S)* **then**
      print G, S
  else if $n_k = 0$ **then**
    CounterExamplesIn(G,$\{A_1^{n_1}, \ldots, A_{k-1}^{n_{k-1}}\}$,$S$,$H$)
  else if A_k *contains a single element* a **then**
    CounterExamplesIn(G, $\{A_1^{n_1}, \ldots, A_{k-1}^{n_{k-1}}\}$, $S \cup \{a^{n_k}\}$, H)
  else
    CounterExamplesWith(G,$\{A_1^{n_1}, \ldots, A_{k-1}^{n_{k-1}}\}$,$S$,$H$,$A_k$, n_k)

Function *CounterExamplesWith(G,$\{A_1^{n_1}, \ldots, A_k^{n_k}\}$,S,H,A,n)*
  `// Helper function that iterates through subsets of` A
  for $1 \leq h \leq n$ **do**
    $X \leftarrow \{B \subset A \mid |B| = h\}$
    for *one* B *in each* H*-orbit of* X **do**
      $K \leftarrow$ stabilizer of B for action of H on X
      CounterExamplesIn(G, $\{A_1^{n_1}, \ldots, A_k^{n_k}, B^{n-h}\}$,$S \cup B$,$K$)

This is clearly faster than a plain application of Algorithm 4, because an r-fold iteration is replaced by an $(r-1)$-fold iteration. Notice, however, that the same counter-example can appear more than once in the output, if this method is applied to cases where m_j has multiplicity greater than one, say $m_j = m_{j+1}$. Indeed, a counter-example $(g_1, \ldots, g_r)$ can be obtained by completing two different short sequences, namely $(g_1, \ldots, \widehat{g_j}, \ldots, g_r)$ and $(g_1, \ldots, \widehat{g_{j+1}}, \ldots, g_r)$. If the two short sequences lie in different $\mathrm{Aut}(G)$-orbits, the output will contain two counter-examples in the $\mathrm{Aut}(G)$-orbit of $(g_1, \ldots, g_r)$.

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